L04 Jan13 Base

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Let X be a nonempty set

The smallest possible topology is $\{\phi, X\}$ Indiscrete

The largest possible one is $\mathcal{P}(X)$ Discrete

Qu. What are in between?

Suppose we add subsets one by one

$\{\phi, A, X\}$ for some $\phi \subsetneqq A \subsetneqq X$ is a topology

What about $\{\phi, A, B, X\}$, $\phi \subsetneqq A \neq B \subsetneqq X$?

Obviously, we need $\{\phi, A\cap B, A, B, A\cup B, X\}$

Qu. Take $\{\phi, X\} \subsetneqq \mathcal{S} \subsetneqq \mathcal{P}(X)$, how to get from $\mathcal{S}$ to a topology

Clarify: It means minimal one Qu: unique?

Think: If there are several topologies, how will we look for a unique minimal one?

we clearly expect $\bigcap_{\alpha\in I} \mathcal{T}_\alpha$

From the given $\mathcal{S}$, to get a topology, we may add

($\cup_1$) unions of sets in $\mathcal{S}$

($\cap_1$) finite intersections of sets in $\mathcal{S}$

($\cup_{n+1}$) unions of sets in $\cup_n$ & $\cap_n$

($\cap_{n+1}$) finite intersections of sets in $\cup_n$ & $\cap_n$

} and so on!!

Qu: Is there a safe systematic method?

Theorem Given any $\mathcal{S} \subset \mathcal{P}(X)$.
First, take all finite intersections on $\mathcal{S}$;
then, take all arbitrary unions on these sets,
the result is a topology.

$$\mathcal{B} = \{\cap \mathcal{F} : \text{finite } \mathcal{F} \subset \mathcal{S}\}$$

$$\mathcal{T} = \{\cup \mathcal{A} : \text{any } \mathcal{A} \subset \mathcal{B}\}$$ is the smallest topology containing $\mathcal{S}$

It is called *the topology generated by $\mathcal{S}$*.
We say that $\mathcal{S}$ is a *subbase (subbasis)* of the topology.

Standard Topology for $\mathbb{R}$ can be generated by $\mathcal{S} = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$

Lower Limit Topology for $\mathbb{R}$ is generated by $\mathcal{B} = \{[a, b) : a < b \in \mathbb{R}\}$

Note $\mathcal{B}$ is special that finite intersections on it will not produce additional sets.

Definition Let $\mathcal{B} \subset \mathcal{P}(X)$. If $\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$ is a topology for X, then $\mathcal{B}$ is called *a base (basis) of $\mathcal{T}$*.

Fact. Starting from $\mathcal{S} \subset \mathcal{P}(X)$

1. $\mathcal{S}$ always generates a topology, i.e. it is always a subbase of a topology
2. But, we do not know if it is a base i.e. additional requirement is needed.

Qu: Any suggestion of the requirement?

Hint. About finite intersection!

Proposition. If the two conditions are satisfied

1. $\phi, X \in \mathcal{S}$
2. $\forall\ U, V \in \mathcal{S}\ \forall\ x \in U \cap V, \exists\ W \in \mathcal{S}\ \ x \in W \subset U \cap V$

then $\mathcal{S}$ is indeed a base for a topology, i.e.,

$$\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{S}\}$$

Sketch The crucial step is about intersection

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{S}$, need to prove

$$(\cup \mathcal{A}_1) \cap (\cup \mathcal{A}_2) \in \mathcal{T}$$

Take $x \in (\cup \mathcal{A}_1) \cap (\cup \mathcal{A}_2)$, $\exists\ S_1 \in \mathcal{A}_1, S_2 \in \mathcal{A}_2$ such that $x \in S_1 \cap S_2$ ← may not $\in \mathcal{S}$

By condition 2, $\exists\ S_x \in \mathcal{S}$ such that

$$x \in S_x \subset S_1 \cap S_2$$

Let $\mathcal{C} = \{S_x : x \in (\cup \mathcal{A}_1) \cap (\cup \mathcal{A}_2)\} \subset \mathcal{S}$

Then $\cup \mathcal{C} = (\cup \mathcal{A}_1) \cap (\cup \mathcal{A}_2)$

From this proposition, we see why
finite $\cap$ then arbitrary $\cup$ works.

Proposition (another definition)

A set $\mathcal{B} \subset \mathcal{T}$ is a base $\Longleftrightarrow$
$\forall x \in G \in \mathcal{T}$ $\exists B \in \mathcal{B}$ with $x \in B \subset G$

Qu. How to prove it?
The harder part is "$\Longleftarrow$".

Qu. Why do base and subbase important?

Example. Not using metric, how do we
describe the open sets of standard $\mathbb{R}$?
Later, we'll see more use of subbase.

Let $\mathcal{T}_1$ and $\mathcal{T}_2 \subset \mathcal{P}(X)$ are topologies on X

Qu. Can we compare them?
Obviously, can ask $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_1 \supset \mathcal{T}_2$!

But, this may not be easy if we only
know bases $\mathcal{B}_1 \subset \mathcal{T}_1$ and $\mathcal{B}_2 \subset \mathcal{T}_2$

Fact. $\mathcal{T}_1 \subset \mathcal{T}_2 \Longleftrightarrow$
$\forall x \in B_1 \in \mathcal{B}_1$ $\exists B_2 \in \mathcal{B}_2$
such that $x \in B_2 \subset B_1$

Note. That why we call $\mathcal{T}_2$ is finer

A topological space $(X, \mathcal{T})$ is 2nd countable if it has a countable base

Notation. C_{II} or 2nd-N

A nbhd base / Local base at $x \in X$ is a set $\mathcal{U}_x$ of nbhds at x such that $\forall$ nbhd W of x $\exists$ $U \in \mathcal{U}_x$ such that $x \in U \subset W$

Example. On any metric space (X, d)

$$\underbrace{\mathcal{U}_x = \{B(x, \tfrac{1}{n}) : 1 \le n \in \mathbb{N}\}}_{\text{countable local base}}$$

Definition. A topological space $(X, \mathcal{T})$ is 1st countable if every $x \in X$ has a countable local base.

Qu. $C_{II} \Rightarrow C_I$?? Why?

Trivial: At any $x \in X$, a base $\mathcal{B}$ always determines a local base there. (Just throw away some)

Example. $(\mathbb{R}^n, d)$ has a countable base

$$\mathcal{B} = \{B(q, \tfrac{1}{n}) : q \in \mathbb{Q}^n, 1 \le n \in \mathbb{N}\}$$

Key reason: $\overline{\mathbb{Q}^n} = \mathbb{R}^n$

Definition. $(X, \mathcal{T})$ is called separable if $\exists$ countable $D \subset X$ such that $\underbrace{\overline{D} = X}_{D \text{ is a dense subset}}$

We have several concepts

C_{II}, C_I, separable

Known: $C_{II} \Rightarrow C_I$

Example: $\mathbb{R}^n$, separable & metric ($\Rightarrow C_I$)

Expect it is C_{II}

Qu: $C_{II} \overset{?}{\Rightarrow}$ separable

Qu: C_I & separable $\overset{?}{\Rightarrow} C_{II}$

Proposition $C_{II} \Rightarrow$ Separable

We have a countable base

$$\mathcal{B} = \{ B_j : j \in \mathbb{N} \}$$

Need $D \subset X$ that is countable, $\overline{D} = X$

How to construct it from $\mathcal{B}$?

Naturally, take one point from each, $x_j \in B_j$

$$D = \{ x_j : j \in \mathbb{N} \}$$ Obviously countable

Why dense? $\overline{D} = X$ ← logical statement

$$\forall x \in G \in \mathcal{J} \quad G \cap D \neq \phi$$

Take $G \in \mathcal{J}$, by that $\mathcal{B}$ is a base

$$\exists \; B_j \subset G \quad \therefore x_j \in G \cap D \neq \phi$$